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Finite Difference Method for Nonlocal Singularly Perturbed Problem

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Abstract: *In this study, nonlinear singularly perturbed problems with nonlocal condition are evaluated by finite difference method. The exact solution $u(x)$ has boundary layers at $x = 0$ and $x = 1$. We present some properties of the exact solution of the multi-point boundary value problem (1)-(3). According to the perturbation parameter, by the method of integral identities with the use exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form uniformly convergent finite difference scheme on Bakhvalov mesh is established. The error analysis for the difference scheme is performed. $\varepsilon -$ uniform convergence for approximate solution in the discrete maximum norm is provided, which is the first-order (O(h)). This theoretical process is applied on the sample. By Thomas Algorithm, it has been shown to be consistent with the theoretical results of numerical results. The results were embodied in table and graphs. The relationship between the approximate solution with the exact solution are obtained by Maple 10 computer program.*

Keywords: *Singular perturbation, finite difference method, Bakhvalov mesh, nonlocal boundary condition, uniform convergence.*

1. Introduction

In this paper we shall consider nonlocal singularly perturbed nonlinear problem

$$-\varepsilon^2 u'' + f(x,u) = 0, \quad 0 \le x \le 1, \tag{1}$$

$$u(0) = 0 \tag{2}$$

$$k_0 u(1) = \sum_{i=1}^{m} k_i u(s_i) + k_{m+1} \int_0^1 u(x)dx + d \tag{3}$$

$$f(x,u) \in C^1\,([0,1]xR), \qquad \frac{\partial f}{\partial u}(x,u) \ge \alpha > 0, \;\; s_i \in (0,1), \qquad i = 1,\ldots,m, \;\; k_0 \ge 0$$

where, $0 < \varepsilon \ll 1$ is small perturbation parameter.

Singularly perturbed differential equations are symbolized by the existence of a small parameter $\varepsilon -$ multiplying the highest order derivatives [4]. Like these problems, thin transition layers where the solution varies rapidly, while away from layers it behaves regularly and varies slowly. Classical numerical methods are inappropriate for the singularly perturbed. So, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value ε; that is, methods that are convergence ε uniformly [18]. Such problems arise in many areas of applied mathematics: fluid mechanics, chemical-reactor theory, control theory, electrical networks, and other physical models. In recent years, Singularly perturbed differential equations were studied by many authors in various fields of applied mathematics and engineering. For examples, Cziegis [1] studied the numerical solution of singularly perturbed nonlocal problem. Cziegis [2] analyzed the difference schemes for problems with nonlocal conditions. Bakhvalov [3] investigated on optimization of methods for solving boundary-value problems in the presence of a boundary layer. Amiraliyev and Cakir [5] studied numerical solution of a singularly perturbed three -point boundary value problem. Amiraliyev and Çakır [6] researched numerical solution of the singularly perturbed problem with nonlocal boundary condition. Amiraliyev and Duru [7] studied a note on a parameterized singular perturbation problem. Boglaev [8] applied uniform numerical methods on arbitrary meshes for singularly perturbed problems with discontinuous data. Adzic and Ovcin [9] obtained nonlinear spp with nonlocal boundary conditions and spectral approximation. Amiraliyev, Amiraliyeva and Kudu [13] analyzed a numerical treatment for singularly perturbed differential equations with integral boundary condition. Herceg [10] estimated on the numerical solution of a singularly perturbed nonlocal problem. Herceg [11] analyzed solving a nonlocal singularly perturbed problem

by splines in tension. Çakır [14] researched uniform second –order difference method for a singularly perturbed three-point boundary value problem. Geng [12] gave a numerical algorithm for nonlinear multi-point boundary value problems. Çakır and Arslan [15] applied a numerical method for nonlinear singularly perturbed multi-point boundary value problem. Bitsadze and Samarskii [16] studied on some simpler generalization of linear elliptic boundary value problems. Çakır [17] obtain a numerical study on the difference solution of singularly perturbed semilinear problem with integral boundary condition.Çakır [5] applied the difference method on a Shishkin mesh to the singularly perturbed three-point boundary value problem.

Some methods for singularly perturbed problems have been studied in different ways [9-12]. In this study we present uniformly convergent difference scheme on an equidistant mesh for the numerical solution of the problem (1)–(3). The difference scheme is constructed by the method integral identities with the use exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form [18]. In Section 2, we obtain some analytical results of the multi-point bounday value problem (1)-(3). The difference scheme constructed on Bakhvalov mesh for numerical solution (1)–(3) is described in Section 3. We prove that the method is first–order convergent in the discrete maximum norm. In the following Section, a numerical example is considered and constructed which is in agreement with the theoretical results. According to results, the finite difference method on Bakhvalov mesh is more powerful method than other methods for nonlinear singularly perturbed multi-point boundary value problem.

2. The continuous problem

Here we give some properties of the solution of (1)-(3) with Lemma 2.1. Because, they are needed in later sections for the analysis appropriate numerical solution. We use $\|g\|_\infty$for the continuous maximum norm on the [0,1], where $g(x)$, is any continuous function.

Lemma 2.1.

Let $f(x,u) \in C^1([0,1]xR)$ and $\frac{\partial f}{\partial u}(x,u)$ is uniformly bounded in $u(x)$. We assume that

$$\sum_{i=1}^{m} k_i w_0(s_i) + k_{m+1}\int_0^1 u(x)dx < k_0 \tag{4}$$

where $w_0(x) \geq |w(x)|$,

$$-\varepsilon^2 w'' + a(x)w(x) = 0 \tag{5}$$

$$w(0) = 0, \qquad w(1) = 1 \tag{6}$$

the solution of this problem.

So, the solution of the Eq. (1) satisfies the inequalities

$$\|u(x)\|_{C[0,1]} \leq C_0 \tag{7}$$

and

$$|u'(x)| \leq C\left\{1 + \frac{1}{\varepsilon}\left[e^{\frac{-\sqrt{\alpha}x}{\varepsilon}} + e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right\}, 0 < x < 1 \tag{8}$$

where, C_0 and C are constants independent of ε and the h.

Proof. We use intermediate value theorem for $f(x,u)$ in the Eq.(1) and we obtain the following the equalities,

$$\frac{f(x,u) - f(x,0)}{u - 0} = \frac{\partial f}{\partial u}(x,\vartheta), \qquad \vartheta = \gamma u, \qquad 0 < \gamma < 1$$

$$f(x,u) = f(x,0) + u(x)\frac{\partial f}{\partial u}(x,\vartheta)$$

$$= -F(x) + a(x)u(x)$$

where $a(x), F(x)$ are sufficiently smooth on $[0,1]$ and $[0,1]xR$.

Also,

$$\frac{\partial f}{\partial u}(x,\vartheta) = a(x) \geq \alpha > 0, \qquad \vartheta - intermediate\ value.$$

So, we obtain the following linear equation,

$$-\varepsilon^2 u'' + a(x)u(x) = F(x), \qquad 0 < x < 1. \tag{9}$$

Now, we can write the solution of the Eq.(9) as follows

$$u(x) = v(x) + \lambda w(x) \tag{10}$$

$$-\varepsilon^2 v'' + a(x)v(x) = F(x) \tag{11}$$

$$v(0) = 0, \qquad v(1) = 0 \tag{12}$$

where $v(x)$ is solution of the Eq. (11)- the Eq. (12).

First, we give the estimate $v(x)$,

$$|v(x)| \leq |v(0)| + |v(1)| + \alpha^{-1}\|F\|_\infty \leq C_1. \tag{13}$$

Second, we give the estimate $w(x)$,

$$|w(x)| \leq |w(0)| + |w(1)| + \alpha^{-1}\|0\|_\infty \leq 1 \tag{14}$$

Then, from the Eq.(13) - the Eq.(14) we have the following inequality

$$|u(x)| \leq |v(x)| + |\lambda||w(x)| \leq C_1 + |\lambda|$$

$$|u(x)| \leq C_0\,. \tag{15}$$

we now prove the estimate the Eq. (8).

if $u''(x)$ is pulled from the Eq. (9), we obtain

$$u''(x) = \frac{-1}{\varepsilon^2}[F(x) - a(x)u(x)] \tag{16}$$

and from the Eq. (16)

$$|u''(x)| \le \left|\frac{1}{\varepsilon^2}[F(x) - a(x)u(x)]\right| \le \frac{C}{\varepsilon^2}. \tag{17}$$

Now, we take derivative of the Eq. (9) and if it called $u'(x) = v_0(x)$,

the Eq. (9) takes the form with boundary condition

$$-\varepsilon^2 v_0'' + a(x)v_0(x) = \emptyset(x) \tag{18}$$

$$u'(0) = v_0(0) = 0, u'(1) = v_0(1) = 1 \tag{19}$$

In a similar manner, we proceed with the estimation of $\emptyset(x), u'(0)$ and $u'(1)$ respectively,

from the Eq. (7), we obtain the following equation

$$\emptyset(x) = F'(x) - a'(x)u(x) \le C_1. \tag{20}$$

We use the relation for $g \in C^2$ as follows:

$$g'(x) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} - \int_\alpha^\beta \left[\frac{\beta - t}{\beta - \alpha} - T_0(x - t)\right] g''(t)dt, \alpha < x < \beta, \qquad \alpha < \beta \tag{21}$$

where

$$T_0(x - t) = \begin{cases} 1, & x - t > 0 \\ 0, & x - t < 0. \end{cases}$$

the Eq. (21) with the values $g(x) = u(x),\ \alpha = 0,\ \beta = \varepsilon,\ x = 0$ and from the Eq. (7)- the Eq. (17)

$$|u'(0)| \le \left|\frac{u(\varepsilon) - u(0)}{\varepsilon}\right| + \int_0^\varepsilon |u''(t)|dt \le \frac{C}{\varepsilon}. \tag{22}$$

Similarly,

the Eq. (21) with the values $g(x) = u(x),\ \alpha = 1 - \varepsilon,\ \beta = 1,\ x = 1$ and from the Eq. (7)- the Eq. (17)

$$|u'(1)| \le \left|\frac{u(1) - u(1 - \varepsilon)}{\varepsilon}\right| + \int_{1-\varepsilon}^1 |u''(t)|dt \le \frac{C}{\varepsilon}. \tag{23}$$

We write the solution of the Eq. (18)- the Eq.(19) in the form,

$$v_0(x) = v_1(x) + v_2(x)$$

where $v_1(x),\ v_2(x)$ are respectively the solution of the following problems,

$$-\varepsilon^2 v_1''(x) + a(x)v_1(x) = \emptyset(x) \tag{24}$$

$$v_1(0) = 0, u'(1) = v_1(1) = 0 \tag{25}$$

$$-\varepsilon^2 v_2{}''(x) + a(x)v_2(x) = 0, \tag{26}$$

$$v_2(0) = |v_0(0)|, u'(1) = v_2(1) = |v_0(1)|$$

According to the maximum principle in the Eq.(24)- the Eq. (25),we can give the following Barrier function,

$$\psi_1(x) = \mp v_1(x) + \alpha^{-1}\|\emptyset(x)\|_\infty$$

This Barrier function provides the conditions of the maximum principle and

$$v_1(x) \le C.$$

In a similar manner, according to the maximum principle in the Eq.(26), we can write

$$v_2(x) \le \theta(x)$$

where $\theta(x)$ is the solution of the following problem with constant coefficient,

$$-\varepsilon^2 \theta''(x) + \alpha\theta(x) = 0 \tag{27}$$

$$\theta(0) = |v_0(0)| \le \frac{C}{\varepsilon}\ , \qquad \theta(1) = |v_0(1)| \le \frac{C}{\varepsilon} \tag{28}$$

where $a(x) \ge \alpha > 0$ and the solution of $\theta(x)$ as follows,

$$\theta(x) = \frac{v_0(0)\left[e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}} - e^{\frac{\sqrt{\alpha}(1-x)}{\varepsilon}}\right] + v_0(1)\left[e^{\frac{-\sqrt{\alpha}x}{\varepsilon}} - e^{\frac{\sqrt{\alpha}x}{\varepsilon}}\right]}{\left[-e^{\frac{\sqrt{\alpha}}{\varepsilon}} + e^{\frac{-\sqrt{\alpha}}{\varepsilon}}\right]} \tag{29}$$

after some arragement, we can obtain,

$$\theta(x) \le \frac{C}{\varepsilon}\left[e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]. \tag{30}$$

Finally, from $u'(x) = v_0(x)$, $|v_1(x)| \le C$, $\theta(x) \le v_2(x)$, $v_0(x) = v_1(x) + v_2(x)$, we have the following inequality,

$$\begin{aligned} |u'(x)| &= |v_0(x)| \le |v_1(x)| + |v_2(x)| \\ &\le C + \frac{C}{\varepsilon}\left[e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right] \le C\left\{1 + \frac{1}{\varepsilon}\left[e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{-\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right\} \end{aligned}$$

which leads to the Eq. (8).

3. Costruction of the Difference Scheme and Non-Unifom Mesh

In what follows, we give by the following non-uniform mesh on [0,1]:

$$\overline{\omega}_N = \left\{0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, x_{N_i} = s_i, N_i = \frac{s_i . N}{1}\right\}.$$

We present some properties of the mesh function $g(x)$ defined on $\overline{\omega}_N$, which is needed in this section for analysis of the numerical solution.

$$g_{x,i} = \frac{g_{i+1} - g_i}{h_{i+1}},$$

$$g_{\bar{x},i} = \frac{g_i - g_{i-1}}{h_i}, \qquad g_{\hat{x},i} = \frac{g_{i+1} - g_i}{\hbar_i}$$

$$g_{\bar{x}\hat{x},i} = \frac{g_{x,i} - g_{\bar{x},i}}{\hbar_i} = \frac{1}{\hbar_i}\left(\frac{g_{i+1} - g_i}{h_{i+1}} - \frac{g_i - g_{i-1}}{h_i}\right)$$

$$h_i = x_i - x_{i-1}, \; \hbar_i = \frac{h_i + h_{i+1}}{2}$$

$$\|g\|_\infty = \|g\|_{\infty, \; \bar{\omega}_N} = \max_{0 \le i \le N} |g_i|$$

The difference scheme, we shall construct as following from the identity

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} -\varepsilon^2 u''(x)\varphi_i(x)dx + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i(x)dx = 0, \qquad i = \overline{1, N-1} \tag{31}$$

where $\{\varphi_i(x)\}$, $i = \overline{1, N-1}$ are the basis functions and having the form

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \dfrac{x - x_{i-1}}{h_i} & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) = \dfrac{x_{i+1} - x}{h_{i+1}} & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}) \end{cases}$$

$\varphi_i^{(1)}$ (x) and $\varphi_i^{(2)}(x)$ respectively, are the solutions of the following problems,

$$-\varepsilon^2 \varphi'' = 0 \tag{32}$$

$$\varphi(x_{i-1}) = 0, \varphi(x_i) = 1 \tag{33}$$

$$-\varepsilon^2 \varphi'' = 0 \tag{34}$$

$$\varphi(x_i) = 1, \varphi(x_{i+1}) = 0. \tag{35}$$

If we rearrange the Eq. (31) it gives

$$-\varepsilon^2 \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} u''(x)\varphi_i(x)dx + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i(x)dx = 0, \qquad i = \overline{1, N-1} \tag{36}$$

and after a sample calculation

$$\varepsilon^2 \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x)dx + f(x_i, u_i) + R_i = 0, \qquad i = \overline{1, N-1} \tag{37}$$

where

$$R_i = \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \varphi_i(x) \int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx} f(x,u) K_0^*(x,\xi) d\xi, \qquad i = \overline{1, N-1} \tag{38}$$

and

$$K_0^*(x,\xi) = T_0(x-\xi) - T_0\left(\frac{x_{i+1}-x_{i-1}}{2} - \xi\right) + (x_{i+1}-x_{i-1})^{-\mathbf{1}}(x_{i+1}-\xi)^0\left(\frac{x_{i+1}-x_{i-1}}{2} - x\right).$$

So, from the Eq. (37), we propose the following difference scheme for approximation the Eq. (1)-the Eq. (3):

$$-\varepsilon^2 u_{\bar{x}\hat{x},i} + f(x_i,u_i) + R_i = 0 \qquad i = \overline{1, N-1} \tag{39}$$

It is now necessary to define an approximation for the second boundary condition of the Eq. (1). We accepted that x_{N_i} is the mesh point nearest to s_i.

$$k_0 u_N = \sum_{i=1}^{m} k_i u_{N_i} + k_{m+1}\int_0^1 u(x)dx + d$$

$$= \sum_{i=1}^{m} k_i u_{N_i} + k_{m+1}\left[\sum_{i=1}^{N} h_i u_i + r_i\right] + d \tag{40}$$

where remainder term

$$r_i = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(\xi - x_{i-1})\frac{d}{dx}\,u(\xi)d\xi. \tag{41}$$

By neglecting R_i and r_i in the Eq. (39) and the Eq. (40), we may propose the following difference scheme for approximating the Eq. (1)- the Eq. (3):

$$-\varepsilon^2 y_{\bar{x}\hat{x},i} + f(x_i,y_i) = 0\,, \quad i = \overline{1,N} \tag{42}$$

$$y_0 = 0 \tag{43}$$

$$k_0 y_N = \sum_{i=1}^{m} k_i y_{N_i} + k_{m+1}\sum_{i=1}^{N} h_i y_i + d \tag{44}$$

We will use the Bakhvalov mesh to be ε- uniform convergent of the difference scheme the Eq. (42)- the Eq. (44). Transition points of the Bakhvalov mes are defined as

$$\sigma = min\left\{\frac{1}{4}, -\left(\sqrt{\alpha}\right)^{-1}\varepsilon ln\varepsilon\right\}.$$

and x_i mesh points as follows:

$\sigma < \frac{1}{4}$, $x_i \in [0,\sigma]$, $x_i = -\left(\sqrt{\alpha}\right)^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\frac{4i}{N}\right], i = 0,1,\dots,\frac{N}{4}.$

$\sigma = \frac{1}{4}$, $x_i \in [0,\sigma], x_i = -\left(\sqrt{\alpha}\right)^{-1}\varepsilon\ln\left[1-\left(1-e^{\frac{-(\sqrt{\alpha})^{-1}}{4\varepsilon}}\right)\frac{4i}{N}\right], i = 0,1,\dots,\frac{N}{4}$

$x_i \in [\sigma, 1-\sigma], x_i = \sigma + \left(i - \frac{N}{4}\right)h^{(1)},\ i = \frac{N}{4}+1,\dots,\frac{3N}{4}, h^{(1)} = \frac{2(1-2\sigma)}{N}$

$x_i \in [1-\sigma, 1],\, x_i = 1-\sigma - \left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln\left(1-(1-\varepsilon)\frac{4\left(i-\frac{3N}{4}\right)}{N}\right), i = \frac{3N}{4}+1, \dots, N$

$1-\sigma = \frac{3}{4},\ x_i \in [1-\sigma, 1],\ x_i = -\left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln\left[1-\left(1-e^{\frac{-\left(\sqrt{\alpha}\right)^{-1}}{4\varepsilon}}\right)\frac{4\left(i-\frac{3N}{4}\right)}{N}\right]$

$$i = \frac{3N}{4}+1, \dots, N$$

where, N is even number, $\sigma \ll 1$.

4. Uniform Error Estimates

Let $z = y - u,\ x \in \bar{\omega}_N$, which is the error function of the difference scheme the Eq. (42)- the Eq. (44) and the solution of the discrete problem

$$lz \equiv -\varepsilon^2 z_{\bar{x}\hat{x},i} + [f(x_i, y_i) - f(x_i, u_i)] = R_i\,, \quad i = \overline{1, N-1} \tag{45}$$

$$z_0 = 0 \tag{46}$$

$$k_0 z_N = \sum_{i=1}^{m} k_i z_{N_i} + k_{m+1} \sum_{i=1}^{N} h_i z_i = r_i\,, \quad i = \overline{1, N} \tag{47}$$

where R_i and r_i are defined in the Eq.(38) and the Eq.(41).

Lemma 4. 2. The solution z_i of problem the Eq. (45)- the Eq. (47) satisfies following

$$\|R\|_{\infty, \bar{\omega}_N} \le CN^{-1} \tag{48}$$

$$|r|_{\infty, \bar{\omega}_{N_N}} \le CN^{-1}\,. \tag{49}$$

Proof. We shall evaulate reminder term R_i,

1) For $\sigma < \frac{1}{4}, x_i \in [0, \sigma]$

For $x_i = -\beta^{-1} \varepsilon \ln\left[1-(1-\varepsilon)\frac{4i}{N}\right], i = 0,1, \dots, \frac{N}{4}$

we obtain

$$h_i = x_i - x_{i-1} = -\left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln\left[1-(1-\varepsilon)\frac{4i}{N}\right] + \left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln\left[1-(1-\varepsilon)\frac{4(i-1)}{N}\right].$$

According to i, we use the intermediate value theorem as follows:

$$h_i = \left(\sqrt{\alpha}\right)^{-1} \varepsilon \frac{4(1-\varepsilon)N^{-1}}{1-i_1 4(1-\varepsilon)N^{-1}} \le 4\left(\sqrt{\alpha}\right)^{-1}(1-\varepsilon)N^{-1} \le CN^{-1}$$

and we have

$$\hbar_i = \frac{h_i + h_{i+1}}{2} \le CN^{-1}.$$

$$e^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}}-e^{\frac{-\sqrt{\alpha}x_{i+1}}{\varepsilon}}\leq e^{\frac{-\sqrt{\alpha}\left(-(\sqrt{\alpha})^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\frac{4(i-1)}{N}\right]\right)}{\varepsilon}}-e^{\frac{-\sqrt{\alpha}\left(-(\sqrt{\alpha})^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\frac{4(i+1)}{N}\right]\right)}{\varepsilon}}$$

$$\leq e^{\ln\left[1-(1-\varepsilon)\frac{4(i-1)}{N}\right]}-e^{\ln\left[1-(1-\varepsilon)\frac{4(i+1)}{N}\right]}\leq e^{\ln\left[1-(1-\varepsilon)\frac{4(i-1)}{N}\right]}-e^{\ln\left[1-(1-\varepsilon)\frac{4(i+1)}{N}\right]}$$

$$\leq e^{\ln\left[-(1-\varepsilon)\frac{4(i-1)}{N}\right]}-e^{\ln\left[-(1-\varepsilon)\frac{4(i+1)}{N}\right]}\leq CN^{-1}$$

and

$$e^{\frac{-\sqrt{\alpha}(1-x_{i+1})}{\varepsilon}}-\ e^{\frac{-\sqrt{\alpha}(1-x_{i-1})}{\varepsilon}}\leq CN^{-1}$$

We use above inequalities and we can obtain

$$R_i \leq 2Ch_i \leq CN^{-1}.$$

2) $\sigma=\frac{1}{4},\ \ x_i\in[0,\sigma]$

We use the followig equation,

$$h_i=x_i-x_{i-1}=-\beta^{-1}\varepsilon\ln\left[1-\left(1-e^{\frac{-\beta}{4\varepsilon}}\right)\frac{4i}{N}\right]+\beta^{-2}\varepsilon\ln\left[1-\left(1-e^{\frac{-\beta}{4\varepsilon}}\right)\frac{4i}{N}\right]$$

we obtain,

$$h_i=\beta^{-1}\varepsilon\frac{4(1-e^{\frac{-\beta}{4\varepsilon}})N^{-1}}{1-i_1 4(1-e^{\frac{-\beta}{4\varepsilon}})N^{-1}}\leq 4\beta^{-1}N^{-1}\leq CN^{-1}$$

and

$$\hbar_i=\frac{h_i+h_{i+1}}{2}\leq CN^{-1}.$$

After doing some calculation,

$$R_i \leq CN^{-1}\,.$$

3) $x_i\in[\sigma,1-\sigma],\ x_i=\sigma+\left(i-\frac{N}{4}\right)h^{(1)},\ i=\frac{N}{4}+1,\dots,\frac{3N}{4},h^{(1)}=\frac{2(1-2\sigma)}{N},\ x_i\in[\sigma,1-\sigma],$

Let $\sigma=-\beta^{-1}\varepsilon ln\varepsilon<\frac{1}{4}$,

According to i, we use the intermediate value theorem and we obtain,

$$h_i\leq CN^{-1}$$

and

$$\hbar_i=h^{(1)}\leq CN^{-1}.$$

Consequently,

$$R_i\leq CN^{-1}.$$

4) $x_i\in[\sigma,1-\sigma],\sigma=\frac{1}{4}$

$$h_i\leq CN^{-1}$$

and

$$\hbar_i=h^{(1)}\leq CN^{-1}.$$

We have

$$R_i \leq CN^{-1}.$$

$$5)\ xi \in [1-\sigma, 1], \qquad x_i = 1-\sigma+\beta^{-1}\varepsilon \ln\left(1-(1-\varepsilon)\frac{4\left(i-\frac{3N}{4}\right)}{N}\right), \qquad i = \frac{3N}{4}+1, \dots, N$$

where, according to i, we use the intermediate value theorem as follows:

$$h_i = x_i - x_{i-1} = \left(1-\sigma+\beta^{-1}\varepsilon \ln\left(1-(1-\varepsilon)\frac{4\left(i-\frac{3N}{4}\right)}{N}\right)\right)$$
$$-\left(1-\sigma+\beta^{-2}\varepsilon \ln\left(1-(1-\varepsilon)\frac{4\left(i-\frac{3N}{4}\right)}{N}\right)\right)$$

Thus, we obtain

$$\hbar_i = \frac{h_i + h_{i+1}}{2} \leq CN^{-1}$$

and

$$R_i \leq CN^{-1}.$$

$$6)\ 1-\sigma = \frac{3}{4}, \qquad x_i \in [1-\sigma, 1]\,, x_i = -\beta^{-1}\varepsilon \ln\left[1-\left(1-e^{\frac{-\beta}{4\varepsilon}}\right)\frac{4\left(i-\frac{3N}{4}\right)}{N}\right], i = \frac{3N}{4}+1, \dots, N$$

we use the intermediate value theorem and we obtain,

$$\hbar_i = \frac{h_i + h_{i+1}}{2} \leq CN^{-1}.$$

$$R_i = \lambda_i \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Ch_i\varphi_i(x)dx + \lambda_i \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Ch_i\varphi_i(x)dx \leq CN^{-1}$$

After all these calculations, we have

$$\|R\|_{\infty,\bar{\omega}_N} \leq CN^{-1}.$$

$$|r_i| \leq \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(x-x_{i-1})\frac{d}{dx}u(x)dx \leq \sum_{i=1}^{N/4} h^{(1)}\int_{x_{i-1}}^{x_i}\left(1+\frac{1}{\varepsilon}\left[e^{\frac{-\sqrt{\alpha}x}{\varepsilon}}+e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right)dx$$
$$+\sum_{i=N/4+1}^{3N/4} h^{(2)}\int_{x_{i-1}}^{x_i}\left(1+\frac{1}{\varepsilon}\left[e^{\frac{-\sqrt{\alpha}x}{\varepsilon}}+e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right)dx$$
$$+\sum_{i=3N/4+1}^{N} h^{(3)}\int_{x_{i-1}}^{x_i}\left(1+\frac{1}{\varepsilon}\left[e^{\frac{-\sqrt{\alpha}x}{\varepsilon}}+e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right)dx$$

After a sample calculation for r_i, we obtain,

$$r_i \leq CN^{-1}.$$

Lemma 4. 3. Let z_i be solution of the Eq. (45)- the Eq. (47). Then there is the following inequality:

$$\|z\|_{\infty,\bar{\omega}_N} \leq C\left[\|R\|_{\infty,\omega_N} + |r|\right] \tag{50}$$

İspat. If we rearrange the Eq. (45), it gives

$$lz \equiv -\varepsilon^2 z_{\bar{x}\hat{x},i} + a_i z_i = R_i\,, \quad i = \overline{1, N-1}$$

where

$$a_i = \frac{\partial F}{\partial u}(t_i, u_i + \gamma z_i)\,, \qquad 0 < \gamma < 1$$

we can use the maximum principle, and so it is easy to obtain,

$$\|z\|_{\infty,\bar{w}_N} \le |z_N| + \alpha^{-1}\left(\|R\|_{\infty,w_N} + |r|\right)$$

$$\le \left|\sum_{i=1}^{m} k_i z_{N_i} + k_{m+1}\sum_{i=1}^{N} h_i z_i + d - r_i\right| + \alpha^{-1}\left(\|R\|_{\infty,w_N} + |r|\right)$$

$$\le \alpha^{-1}\left(\|R\|_{\infty,w_N} + |r|\right)$$

$$\le C\left[\|R\|_{\infty,w_N} + |r|\right] \tag{51}$$

Theorem 4. 1. Let $u(x)$ be the solution of (1)-(3) and let y be the solution of Eq. (45)- the Eq. (47). Then under the same assumptions of Lemma (4. 2) and Lemma (4.3) the following ε −uniform error estimate

$$\|y - u\|_{\infty,\bar{w}_N} \le CN^{-1}\ln N \tag{52}$$

holds.

5) Algorithm and the Numerical Example

We apply the scheme the Eq. (42)- the Eq. (44) to calculate the solution of approximation of the following problem:

$$-\varepsilon^2 u'' + u^2(x) - f(x) = 0, \tag{53}$$

$$u(0) = 0 \tag{54}$$

$$u(1) = u(0.5) + d \tag{55}$$

which has the exact solution,

$$u(x) = \frac{(2x-1)\left(e^{\frac{1-x}{\varepsilon}} - e^{\frac{x}{\varepsilon}}\right)}{e^{\frac{1}{\varepsilon}} - 1} + 1\,.$$

We solve the difference scheme the Eq. (42)- the Eq. (44) using the following iteration technique,

$$\left[\frac{\varepsilon^2}{\hbar_i h_i}\right] y_{i-1}^{(n)} - \left[\frac{2\varepsilon^2}{h_i h_{i+1}} + \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)\right] y_i^{(n)} + \left[\frac{\varepsilon^2}{\hbar_i h_{i+1}}\right] y_{i+1}^{(n)} (68)$$

$$= f(x_i) + f\left(x_i, y_i^{(n-1)}\right) - y_i^{(n-1)}\frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right), i = 1, \dots, N-1, \qquad n = 1.2, \dots$$

$$y_0^{(n)} = 0, \quad k_0 y_N^{(n)} = \sum_{i=1}^{m} k_i y_{N_i}^{(n-1)} + k_{m+1}\sum_{i=1}^{N} h_i y_i^{(n-1)} + d \tag{56}$$

$$\left[\frac{\varepsilon^2}{\hbar_i h_i}\right] y_{i-1}^{(n)} - \left[\frac{2\varepsilon^2}{h_i h_{i+1}} + \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)\right] y_i^{(n)} + \left[\frac{\varepsilon^2}{\hbar_i h_{i+1}}\right] y_{i+1}^{(n)} \tag{57}$$

$$= f(x_i) + f\left(x_i, y_i^{(n-1)}\right) - y_i^{(n-1)} \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right), i = 1, \dots, N-1,$$

$$y_{\frac{N}{2}}^{(n)} = \mu_{n-1}, \qquad y_N^{(n)} = \mu_{n-1} - 1, \qquad \mu_0 = C_0 \geq 1$$

where

$$\mu_n = \frac{\left(\frac{\varepsilon}{h_2}\right)^2 y_{\frac{N}{2}-1,n} + \left(\frac{\varepsilon}{h_2}\right)^2 y_{\frac{N}{2}+1,n} - 1 + \left(y_{\frac{N}{2}+1,n-1}\right)^2}{2\left(\frac{\varepsilon}{h_2}\right)^2 + y_{\frac{N}{2},n-1}}$$

The system of the (57) is solved by the following procedure,

$$A_i = \frac{\varepsilon^2}{\hbar_i h_i}, \qquad B_i = \frac{\varepsilon^2}{\hbar_i h_{i+1}}, \qquad C_i = \frac{2\varepsilon^2}{h_i h_{i+1}} + \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right),$$

$$F_i = -f\left(x_i, y_i^{(n-1)}\right) + y_i^{(n-1)} \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)$$

$$\alpha_1 = 0, \qquad \beta_1 = 0$$
$$\alpha_{\frac{N}{2}+1} = 0, \qquad \beta_{\frac{N}{2}+1} = \mu_{n-1}$$

$$\alpha_{i+1} = \frac{B_i}{C_i - A_i\alpha_i}, \quad \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - A_i\alpha_i}, \qquad i = 1, \dots, N-1$$

$$y_i^{(n)} = \alpha_{i+1} y_{i+1}^{(n)} + \beta_{i+1}, \qquad y_i^{(0)} = 0.5\,, \qquad i = N-1, \dots, 2, 1.$$

It is easy to verify that

$$A_i > 0,\ B_i > 0\,, C_i > A_i + B_i, \qquad i = 1, \dots, N.$$

For this reason, the described procedure above is stable. Also, the Eq. (42)- the Eq. (44) has only one solution.

In the computations in this section, we will take $d = -1$.

The initial guess in the iteration procedure is $y_i^{(0)} = 0.5$. The stopping criterion is taken as

$$\max_i \left| y_i^{(n+1)} - y_i^{(n)} \right| \leq 10^{-5}.$$

The error estimates are denoted by

$$e_\varepsilon^N = \|y - u\|_{\infty, \bar{\omega}_N}$$

and

$$e^N = \max_{\varepsilon} e_\varepsilon^N.$$

The corresponding ε −uniform convergence rates are computed using the formula

$$P_\varepsilon^N = \log_2\left(\frac{e^N}{e^{2N}}\right).$$

The numerical results obtained from the problem of the difference scheme by comparison, the error and uniform rates of convergence were obtained and these are presentedin Table 1.

Table 1. The computed maximum pointwise errors e^N and e^{2N}, the numerical rate of convergence P^N on the Bakhvalov mesh $\overline{\omega}_N$ for different values of N and ε.

ε/N	8	16	32	64	128	256	512	1024
2^{-10}	0.1844643708 p=0.826	0.1031176337 p=0.973	0.0525233947 p=0.991	0.0264140654 p=1.006	0.0131517954 p=1.025	0.0064588555 p=1.026	0.0031701248 p=1.341	0.0012507942
2^{-12}	0.2186063415 p=0.839	0.1221878445 p=0.933	0.0639603910 p=0.930	0.0335687579 p=0.926	0.0176559863 p=0.939	0.0092080361 p=0.966	0.0047118041 p=0.999	0.0023568060
2^{-14}	0.2468061329 p=0.844	0.1374469602 p=0.910	0.0731044491 p=0.892	0.0393749598 p=0.876	0.0214435825 p=0.874	0.0116955952 p=0.887	0.0063238248 p=0.913	0.0033579638
2^{-16}	0.2701179896 p=0.851	0.1497045164 p=0.896	0.0803967387 p=0.869	0.0439999693 p=0.846	0.0244759852 p=0.835	0.0137180219 p=1.117	0.0063238249 p=0.953	0.0032650757
2^{-18}	0.2895653366 p=0.858	00.1596652402 p=0.888	0.0862725672 p=0.854	0.0477070838 p=0.826	0.0268973496 p=0.810	0.0153331458 p=0.806	0.0087696300 p=0.809	0.0050041978
2^{-20}	0.3059909186 p=0.865	0.1679963817 p=0.881	0.0911603753 p=0.846	0.0506828286 p=0.813	0.0288403106 p=0.794	0.0166239446 p=0.786	0.0096388095 p=0.785	0.0055920974

The results show that the convergence rate of the considered scheme is essentially in accord with theoretical analysis.

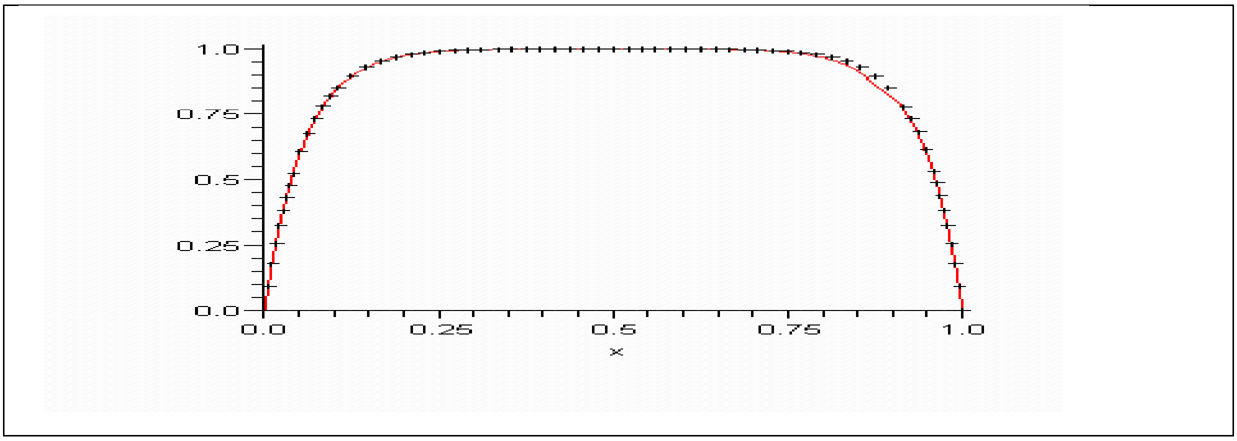

Figure 1. Comparison of Approximate solution and Exact solution for $\varepsilon = 2^{-14}$

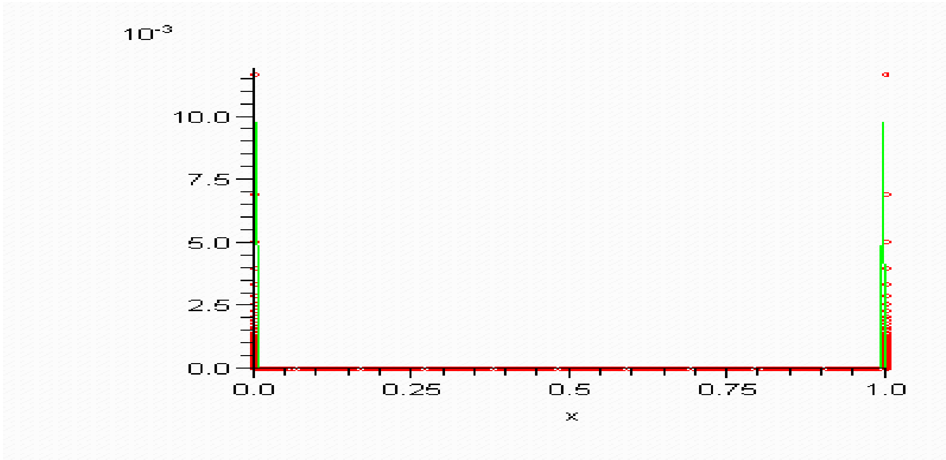

Figure 2. Error distribution for $\varepsilon = 2^{-14}, 2^{-16}, 2^{-18}$ using N=256

From the graps it is show that the error is maximum near the boundary layer and it is almost zero in outer region in the Figure 2. Approximate solution compared with exact solution in Figure 1.

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